Gravitational monopoles with classical torsion

1980 J. Phys. A: Math. Gen. 13 L359

(http://iopscience.iop.org/0305-4470/13/10/003)

J. Phys. A: Math. Gen. **13** (1980) L359–L363. Printed in Great Britain

LETTER TO THE EDITOR

Gravitational monopoles with classical torsion

I M Benn, T Dereli† and R W Tucker
University of Lancaster, Department of Physics, Lancaster, UK

Received 6 March 1980

Abstract. A model is constructed for a space–time geometry that contains Einstein and Stephenson–Yang gravity together with a gravitationally charged vector field in the adjoint representation of the structure group. The model is formulated as an SL(2*C*) gauge theory for a metric connection and a field of orthonormal frames. A simple ansatz renders a solution to the completely coupled system which, it is argued, has the characteristic of a gravitational monopole.

Classical solitons have been discovered in a number of nonlinear field theories. Apart from their intrinsic interest, they are widely expected to play a fundamental role in understanding the full properties of such field theories. Their theoretical existence seems to depend upon inherent nonlinearities of the system, and very often a local gauge covariance is also manifest.

Since gravity may be viewed as a local gauge theory it is natural to enquire about its solitonic content. Indeed, considerable investigation into its Euclidean instanton structure (Hawking 1979) has been carried out. In this Letter we report on a series of classical solutions for a theory of gravity in a Minkowski signatured space–time, some of which closely resemble in analytic form the magnetic monopole solutions of the Georgi–Glashow theory (Georgi and Glashow 1972).

Our basic approach is to regard the theory as endowing a four-dimensional space–time manifold with a series of distinct local structures:

(1) A metric g with signature $(-+++)$.

(2) A metric compatible connection ω.

(3) Four frame fields e^a which are locally orthonormal, i.e. in terms of these 1-forms

$$g=-e^0\otimes e^0+\sum_{k=1}^{3} e^k\otimes e^k.$$

The fundamental metricity condition is ensured by taking ω to be an SL(2*C*) algebra valued 1-form, since this generates the covering group of the Lorentz structure group. Purely gravitational interactions are constructed to be covariant with respect to this connection.

The development of the theory will be expressed using the compact language of the exterior calculus. The analysis has been carried out using the recently developed method of complex quaternionic forms (Tucker 1980), although our results may be readily translated into expressions involving real forms.

† On leave from the Physics Department, Middle East Technical University, Ankara, Turkey.

There has been considerable discussion in the literature about the nature of gravity as a gauge theory. Our view is that any theory with a local gauge invariance is a gauge theory. Whether the connection of the principle bundle is dictated by an action principle (on the base space) or is given by other criteria is largely a matter of detail. For the internal symmetry group an action 4-form, quadratic in the curvatures, is often extremalised for this purpose. For Einstein gravity a different form is adopted, since the metric of the base manifold is also treated as a dynamical degree of freedom.

In the Georgi–Glashow model the magnetic monopole (t'Hooft 1974, Polyakov 1974) is constructed by finding an extremum in the static field energy of a system of coupled isovector Lorentz scalars and SU(2) vector fields. For an analogous system of SL(2C) vector fields coupled to other fields in an inherently curved space, one must seek to isolate potential solitons by other means, since the local energy density including the gravitational contributions is not covariantly defined. It is also important to treat the connection and metric variations as independent, and derive separate systems of equations for their dynamical determination. This will naturally lead to systems of Lorentz frames that may have non-zero torsion. Whether or not the classical torsion has observable effects (besides the soliton) of course depends upon the nature of its coupling to other phenomena (Hehl *et al* 1976).

In Einstein gravity it is customary to identify the weak field disturbances from the Minkowski metric as gravitons. These will be distinguished from the vector fields associated with the propagating connection. We are aware of some of the difficulties associated with quantising these degrees of freedom with certain actions, and will consequently restrict ourselves in this paper to classical considerations.

The theory that we investigate is described in terms of a connection ω, frame e and field ϕ that are fixed by extremalising the action density

$$\Lambda(e, \omega, \phi) = \mathrm{Re}\, S[\rho \wedge *\rho] \tag{1}$$

where

$$\rho = \boldsymbol{R} + \mathrm{i}{*}\boldsymbol{D}\phi + \mu e \wedge \bar{e} \tag{2}$$

and $\boldsymbol{R}$ is the SL(2C) curvature 2-form. The field ϕ is a complex q-vector valued 1-*form* (with 6×4 conventional components $\phi^{ab}{}_{\mu}$) transforming under the adjoint representation of SL(2C) as $\phi \to Q\phi\bar{Q}$, and $\boldsymbol{D}\phi = \mathrm{d}\phi + 2V(\omega \wedge \phi)$ is its exterior covariant derivative. The $*$ defines the usual Hodge dual with respect to the space–time metric. In terms of real forms

$$\Lambda = \rho^{ab} \wedge *\rho_{ab}, \qquad a, b = 0, 1, 2, 3, \tag{3}$$

$$\rho^{ab} = R^{ab} + \tfrac{1}{2}\epsilon^{ab}{}_{cd}{*}\boldsymbol{D}\phi^{cd} + \mu{*}(e^a \wedge e^b). \tag{4}$$

We choose this action since up to a local exact form it expands as

$$\Lambda_1 = \mathrm{Re}\, S(\boldsymbol{R} \wedge *\boldsymbol{R} + \boldsymbol{D}\phi \wedge *\boldsymbol{D}\phi + 2\mathrm{i}\mu \boldsymbol{R} \wedge e \wedge \bar{e} - 2\mathrm{i}\mu \boldsymbol{D}\phi \wedge e \wedge \bar{e} + \mathrm{i}\mu^2 e \wedge \bar{e} \wedge e \wedge \bar{e}). \tag{5}$$

Thus we can identify contributions from the Einstein and Stephenson–Yang (Stephenson 1958, Yang 1974) action together with terms describing the active coupling of the ϕ field to the metric and connection. The last term, with arbitrary real constant μ, is a cosmological one. For vanishing ϕ the theory admits the Schwarzschild solution (with μ) and will accommodate the classical tests. The classical field equations for the connection may be written

$$\boldsymbol{D}{*}\rho = 2\mathrm{i}V(\phi \wedge \rho) \tag{6}$$

or

$$D{*}R = -2V(\phi \wedge {*}D\phi) + 2i\mu V(\phi \wedge e \wedge \bar{e}) - i\mu D(e \wedge \bar{e}). \tag{7}$$

From ϕ variations one obtains

$$D\rho = 0 \tag{8}$$

or

$$D{*}D\phi = i\mu D(e \wedge \bar{e}). \tag{9}$$

Since the source term here may be written $2i\mu V(T \wedge \bar{e})$ in terms of the torsion T, we may identify ϕ as being partially generated by the source of torsion in the theory. Finally, by varying the orthonormal frames, the Einstein equation may be expressed as

$$\mathcal{H}[4i\mu\bar{e} \wedge ({*}\rho)] = i\tau \tag{10}$$

where the anti-Hermitian 3-form τ has components

$$\tau_\alpha = \mathrm{Re}\, S[i_{X_\alpha}\rho \wedge {*}\rho + \rho \wedge i_{X_\alpha}{*}\rho - 2i\rho \wedge i_{X_\alpha}D\phi - 2i i_{X_\alpha}({*}D\phi) \wedge {*}\rho],$$
$$\alpha = 0, 1, 2, 3, \qquad i_{X_\alpha}(e^\beta) = \delta^\beta_\alpha. \tag{11}$$

Writing the classical field equations (6), (8), (10) in terms of the complex 2-form ρ enables us to locate a solution to this coupled set given by $\rho = 0$. Since $\Lambda_1 = \Lambda - 2d[\mathrm{Im}\, S(R \wedge \phi)]$ and Λ vanishes for this solution, the action Λ_1 becomes a local exact form. By analogy with the SU_2 theory of monopoles in the Georgi–Glashow model, it is tempting to regard ϕ as a kind of SL(2C) Higgs field and $\rho = 0$ as a gravitational Bogomol'ny (1976, Coleman *et al* 1977) condition. We stress however that the theory here is fully covariant and no static or Euclidicity postulate has been made yet. Furthermore R contains both 'magnetic' and 'electric' type curvatures in any frame of reference in general.

To solve the equations

$${*}R - iD\phi + i\mu e \wedge \bar{e} = 0 \tag{12}$$

for the classical degrees of freedom, we shall look for a static spherically symmetric system, where the metric is Minkowskian and R^3 polar coordinates are used in the space-like sections t = constant.

$$e = d(it + rN). \tag{13}$$

In terms of R^3 Cartesians, $r^2 = \Sigma_{i=1}^3 x_i^2$ and $N = \Sigma_{i=1}^3 (x^i/r)\hat{e}_i$ is a field of q-vector normals to spheres about the R^3 origin. (The choice (13) is in fact less general than it need be when $\mu = 0$. Any metric such that $e' = \lambda(r)e$ for a real function λ will also yield the equations (16), (17), (18), (19), reflecting a conformal invariance in the nature of our ansatz (12). The behaviour of the system at infinity is more precisely discussed in terms of a compactified Minkowski (Howe and Tucker 1978) space with such a metric, although in this note we shall be content with (13).) For the other fields a 'dyon'-like ansatz is employed.

$$\omega = \tfrac{1}{2}[K(r) - 1]N\mathrm{d}N - \tfrac{1}{2}i(J(r)/r)N\mathrm{d}t, \tag{14}$$

$$\phi = \tfrac{1}{2}[L(r) - 1]N\mathrm{d}N - \tfrac{1}{2}i(H(r)/r)N\mathrm{d}t, \tag{15}$$

where K, L, J and H are real functions. Inserting (13), (14), (15) into (12) gives the four coupled equations

$$r^2(H/r)' = K^2 - 1 - 4\mu r^2, \tag{16}$$

$$rK' = HK + J(L-1) + 4\mu r^2, \tag{17}$$

$$rL' = KJ - 4\mu r^2, \tag{18}$$

$$r^2(J/r)' = 2K(L-1) + 4\mu r^2. \tag{19}$$

It is observed that if $\mu = 0$ and $J = 0$, $L = 1$ (pure magnetic ansatz for the connection) then the 'magnetic' monopole equations arise with solutions (Prasad and Sommerfield 1975)

$$K_0 = Cr/\sinh Cr, \tag{20}$$

$$H_0 = Cr \coth Cr - 1,$$

where we naturally impose the boundary condition that H/r approach a constant C at large r and $D\phi$ vanishes. Although the action in this limit resembles the Georgi–Glashow SU(2) model in structure, it should be recalled that the ϕ field is a 1-form in the gravitational case. Furthermore, although $de = 0$, since the curvature is non-zero the resulting monopole space–time will have a torsion form

$$T = [K_0(r) - 1]\, d\mathrm{N} \wedge \mathrm{d}r. \tag{21}$$

Asymptotically in r the gauge-invariant field strengths of torsion and curvature vanish, since $K_0(r) \to 0$ and

$$*(T \wedge *\bar{T}) = (2/r^2)(K_0(r) - 1)^2, \tag{22}$$

$$*(R \wedge *R) = \frac{(K_0^2(r) - 1)}{4r^4} + \frac{(K_0'(r))^2}{2r^2}. \tag{23}$$

Since the metric is Minkowskian the space may be termed asymptotically locally flat. The form $\mathrm{Re}\, S[R \wedge *R] = -\mathrm{Im}\, S[R \wedge D\phi]$ in the $\mu = 0$ case and hence yields a finite density $\mathrm{Im}\, S \int_{S^2} R \wedge i_{\partial_t}\phi$ per unit time analogous to the 't Hooft–Polyakov monopole. In the $\mu = 0$ sector of the theory the Einstein tensor contribution to the Einstein equation is absent. However, for small μ one could linearise the equations (16)–(19) with solutions of the form

$$K(r) = K_0(r) + \mu K_1(r) + \ldots,$$

$$H(r) = H_0(r) + \mu H_1(r) + \ldots,$$

$$-1 + L(r) = \mu L_1(r) + \ldots,$$

$$J(r) = \mu J_1(r) + \ldots, \tag{24}$$

and solve them iteratively (on a computer) about $r = 0$.

Besides these monopole-like spaces we observe the following singular solutions (naked singularities?):

(a) $K = 1, \quad L = 1, \quad J = 4\mu r^2, \quad H = -4\mu r^2;$ (25)

(b) $K = 1, \quad L = 1 - 2\mu r^2, \quad J = 0, \quad H = -4\mu r^2;$ (26)

(c) $K = 0, \quad L = 1, \quad J = 0, \quad H = 0.$ (27)

The last solution is the analogue of the Wu–Yang monopole.

Conclusion

We have exhibited a model with classical geometries that contain a non-trivial torsion phase and curvatures that are described as a gravitational monopole. The model contains a gravitationally charged field ϕ, inasmuch as it transforms actively under the structure group of the theory. Although we can offer no immediate interpretation of this field, it is tempting to identify it with a gravitational Higgs field. Indeed, one may simulate an SL(2,C) Higgs effect by postulating that $\phi = \langle\phi_0\rangle e^0$ defines a local ground state for some real q-vector ϕ_0 in a frame with a space-like surface orthogonal to the time direction e^0. The SL(2,C) groups is then broken down to the covering rotation group of the space-like triad (e^1, e^2, e^3) in the class of frames defined by the above ground state. For the monopole geometry a further reduction to U(1) is manifest, and our solution explicitly approaches this ground state for large r. Thus one expects that five of the connection vector fields (including ghosts) will dissociate from the field belonging to the surviving symmetry. It may be possible to relate ϕ to the contortion tensor of the space and hence render the theory entirely geometrical, in which case the cosmological term may simulate the necessary Higgs potential. Although the physical significance of the space–times discussed in this model is by no means obvious, we feel that the structure of the model mirrors a number of the features found in the gauge theories of non-gravitational interactions, and consequently may merit further investigation.

References

Bogomol'ny E B 1976 *Sov. J. Nucl. Phys.* **24** 449
Coleman S, Parke S, Neveu A and Sommerfield C M 1977 *Phys. Rev.* D **15** 544
Georgi H and Glashow S L 1972 *Phys. Rev. Lett.* **28** 1494
Hawking S W 1979 in *Recent Developments in Gravitation, Cargèse 1978* ed M Lévy and S Deser (New York: Plenum)
Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 *Rev. Mod. Phys.* **48** 393
't Hooft G 1974 *Nucl. Phys.* B **79** 276
Howe P S and Tucker R W 1978 *Nucl. Phys.* B **138** 73
Polyakov A M 1974 *JETP Lett.* **20** 194
Prasad M K and Sommerfield C M 1975 *Phys. Rev. Lett.* **35** 760
Stephenson G 1958 *Nuovo Cim.* **9** 263
Tucker R W 1980 *J. Math. Phys.* to be published
Yang C N 1974 *Phys. Rev. Lett.* **33** 445